Syntax Literate: Jurnal Ilmiah Indonesia p–ISSN: 2541-0849 e-ISSN: 2548-1398
Vol. 8, No. 8, Agustus 2023

CHARACTERIZATIONS FOR THE FRAKSIONAL INTEGRAL OPERATOR ON CLASSIC MORREY SPACES

Sahat P. Nainggolan
Del Institute of Technology Faculty of Vocational
Email: sahat.nainggolan@del.ac.id

Abstrak

The boundedness of fractional integral operator I_α on $\mathbb{R}$ was introduced for the first time by Hardy G.H and Littlewood J.E (1928). In their evidence proof, Hardy and Littlewood used maximal operator that later known Hardy-Littlewood inequation. They proved that I_α was limited from the Lebesgue's space $L^p(\mathbb{R}^n)$ to the space $L^q(\mathbb{R}^n)$ with $0 \leq \alpha \leq n$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. In 1938, a mathematician C.B Morrey introduced one of space, namely the Morrey's space with notation $L^{p,\lambda}(\mathbb{R}^n)$. This paper will elaborate the Morrey's space and the boundedness of I_α toward the classic Morrey's space by benefitted the Hardy-Littlewodd maximal operator.

Keywords: Boundedness, Fractional Integral Operator, the maximal operator of Hardy Littewood, Minskowski inequation, Holder inequation, Lebesgue's space,

Introduction

In 1886, Marcell Riesz introduced one function operator known as the fractional integral operator I_α that is:

For example f real-valued function on $\mathbb{R}^n$. for a, $0 < \alpha < n$ dan $x \in \mathbb{R}^n$, fractional integral operator I_α defined as follows:

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

Furthermore, this fractional integral operator is often known as the Riesz potential. The problem studied related to the fractional integral operator above is a limitation problem. As is known, the operator T from space X to space Y said to be limited, if any $M > 1$ such that $\|Tx: Y\| \leq M\|x\|$ with $\|x: X\|$ noting norms x in room X. Then the operators T is said to be limited in space X, If T limited from space X to space$X$.

The limitation of an operator is a property that is expected to be met, because this property leads to conditions that are interesting to study. For example when working with differential equations or integral equations, the limitations of an operator can provide an

How to cite:	Sahat P. Nainggolan (2023) Characterizations for the Fraksional Integral Operator on Classic Morrey Spaces, *(8) 8*, http://dx.doi.org/10.36418/syntax-literate.v6i6
E-ISSN:	2548-1398
Published by:	Ridwan Institute

understanding of certain physical phenomena. Meanwhile in the field of computing, computing will be much easier if you work with a limited number of operators.

For the first time the limitations of the fractional integral operator on \mathbb{R} were proved by Hardy G.H and Littlewood J.E (1928). In their proof, Hardy and Littlewood used the maximal (function) operator which became known as the Hardy-Littlewood inequality.

Furthermore, the limitations of the fractional integral operator I_α proved by them in one of the homogeneous spaces, namely from the Lebesgue space $L^p(\mathbb{R}^n)$ to space $L^q(\mathbb{R}^n)$ with $0 \leq \alpha \leq n$ And $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. Connection p And q always used in proof of the limitations of the fractional integral operator. Furthermore, in 1930 the limitations in the Lebesgue space were refined by Sobolev, so that the important result he obtained was called the Hardy-Littlewood-Sobolev inequality. Several years after that, in 1937 N. Wiener reintroduced the maximal operator, but for the case of a higher dimensional Euclidean space.

In 1938, a mathematician named C.B. Morrey introduced one of the well-known spaces to date, namely the Morrey space denoted by $L^{p,\lambda}(\mathbb{R}^n)$ (Lina, 2013 hal 1). This space is often encountered when studying the Schodinger operator and potential theory where the Morrey space is an extension of the Lebesgue space. After Hardy-Littlewood-Sobolev, the limitations of the fractional integral operator I_α further developed by D.R. Adams in the Morrey room. This result was then proven again by Chiarenza-Frasca using the Fefferman-Stein inequality. Chiarenza-Frasca succeeded in proving the limitations of the fractional integral operator from Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to space $L^{q,\lambda}(\mathbb{R}^n)$.

Based on the description above, this paper discusses whether the fractional integral operator has the same limitations in the previous function space, namely the Lebesgue space. $L^q(\mathbb{R}^n)$. Furthermore, what conditions must be met so that the fractional integral operator is confined to the Morrey space.

Research Methods

1. **Morrey Room $L^{q,\lambda}(\mathbb{R}^n)$**

   The Morrey space is the set of all local Lebesgue integrated functions with an expansion value of a finite q-norm. To define a Morrey space, it is necessary to define a local Lebesgue space $L^q_{loc}(\mathbb{R}^n)$. However, before defining the local Lebesgue space, it is necessary to define the Lebesgue space first $L^q(\mathbb{R}^n)$ namely the space that contains functions equipped with a q-norm whose value is up to $\mathbb{R}^n$. According to Kevin (2014:1) Lebesgue room$L^q(\mathbb{R}^n)$ (named after its discoverer, Henry Lebesgue) is a scalable function space which is a natural embodiment of a finite dimensional vector space equipped with norm$-q, \|.\|_q$. Erwin Kreyszig (1978:61) defines a Lebesgue space and says that a Lebesgue space is a Banach space. The following is a definition of a Lebesgue space $L^q(\mathbb{R}^n)$.

Lebesgue Room $L^q(\mathbb{R}^n)$

For $1 \leq q < \infty$*, Lebesgue room* $L^q(\mathbb{R}^n)$ *contains all scalable functions* f *on* $\mathbb{R}^n$ *that fulfills* $\|f\|_q < \infty$*, with,*

$$\|f\|_q = \left(\int_{R^n} |f(x)|^q dx\right)^{\frac{1}{q}}$$

Example: Suppose function $f(x) = \frac{1}{x}\chi\,\mathbb{R}\backslash[-1,1]$ with $q > 1$. Clear that f measurable function. Because any function that is continuous almost everywhere is a measurable function. It is clear that the function is a continuous function on $\mathbb{R}\backslash[-1,1]$, consequently function $f(x) = \frac{1}{x}\chi\,\mathbb{R}\backslash[-1,1]$ with $q > 1$ is a measurable function. Furthermore,

$$\int_{\mathbb{R}^n} |f(x)|^q dx = \int_{\mathbb{R}\backslash[-1,1]} \frac{1}{|x|^q} dx$$

$$= 2\int_1^\infty \frac{1}{x^q}\, dx$$

$$= 2\,\left[\frac{1}{1-q}\, x^{1-q}\right]_1^\infty = 2\left(\frac{1}{q-1}\right) < \infty$$

Clear $f \in L^q(\mathbb{R}^n)$. But if you pay attention, for $q = 1$, $f \notin L^1(\mathbb{R}^n)$, Because

$\int_{\mathbb{R}} |f(x)| dx = 2\,\int_1^\infty \frac{1}{x} dx = 2. \ln x|_1^\infty = \infty$.

As for the local Lebesgue room $L^q_{loc}(\mathbb{R}^n)$ defined as follows:

Local Lebesgue Room $L^q_{loc}(\mathbb{R}^n)$

Local Lebesgue Room $L^q_{loc}(\mathbb{R}^n)$ *with* $1 \leq q < \infty$ *is a space containing all scalable functions* f *that fulfills :*

$$\int_K |f(x)|^q dx < \infty$$

for each compact subset $K \subseteq \mathbb{R}^n$*. If* $f \in L^q_{loc}(\mathbb{R}^n)$*, so* f *is said to be locally integrated in* $L^q(\mathbb{R}^n)$.

Based on definition 1 above, the membership requirements of $L^q(\mathbb{R}^n)$ still fairly 'rough', because it only requires the finiteness of the expression $\int_{\mathbb{R}^n} |f(x)| dx$. Therefore, it is necessary to add one parameter in the hope that it will refine the membership conditions $L^q(\mathbb{R}^n)$. The result of refinement of the Lebesgue space $L^q(\mathbb{R}^n)$ by adding one parameter, it is called a Morrey space $L^{q,\lambda}(\mathbb{R}^n)$ (named after its discoverer, Charles B. Morrey, Jr). In brief, $L^{q,\lambda}(\mathbb{R}^n)$ related to the local properties of $L^q_{loc}(\mathbb{R}^n)$ which is defined on $\mathbb{R}^n$, whereas $L^q(\mathbb{R}^n)$ related to global properties. The following is the definition of a Morrey space $L^{q,\lambda}(\mathbb{R}^n)$.

Morrey Room $L^{q,\lambda}(\mathbb{R}^n)$

For example B(x,r) is an open ball di $\mathbb{R}^n$. n, $L^{q,\lambda}(\mathbb{R}^n)$ *is the set of all functions* $L^q_{loc}(\mathbb{R}^n)$ *that fulfills :*

$$\|f\|_{q,\lambda} = \sup_{B=B(x,r)} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q}} < \infty$$

Where $B(x,r) \subseteq \mathbb{R}^n$ is denotes the ball centered at x and fingers $r > 0$ (Kreyszig, 1978:18).

View shape :

$$\|f\|_{q,\lambda} = \sup_{B=B(x,r)} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q}}.$$

for case $\lambda = 0$ obtained:

$$\|f\|_{q,0} = \sup_{B=B(x,r)} \left(\frac{1}{r^0} \int_{B(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q}}$$

$$= \sup_{B=B(x,r)} \left(\frac{1}{1} \int_{B(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q}}$$

$$= \sup_{B=B(x,r)} \left(\int_{B(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q}}.$$

Because $\int_{B(x,r)} |f(y)|^q \, dy < \infty$ And $B(x,r) \subseteq \mathbb{R}^n$ and the supremum value of the integral is the result of the integral itself, ie:

$$\sup_{B=B(x,r)} \left(\int_{B(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q}} = \left(\int_{R^n} |f(y)|^q \, dy \right)^{\frac{1}{q}} = L^q(\mathbb{R}^n)$$

As a result for cases $\lambda = 0, L^{q,\lambda}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$. This means that the Lebesgue space is a special form of the Morrey space in this case $\lambda = 0$.

By paying attention to the definition given above, the following proposition can be derived :

Proposition 1.
Function f *is a member of* $L^{q,\lambda}(\mathbb{R}^n)$ *if and only if there is* $C > 0$ *such that* $\int_{B(x,r)} |f(x)|^q dx < Cr^{\lambda}$ *for each* $x \in \mathbb{R}^n$ *dan* $r > 0$.

Proof :
For any function $f \in L^{q,\lambda}(\mathbb{R}^n)$ maka $\|f\|_{q,\lambda} < \infty$, Where

$$\|f\|_{q,\lambda} = \sup_{B=B(x,r)} \left(\frac{1}{r^{\lambda}} \int_{B(x,r)} |f(x)|^q \, dx\right)^{\frac{1}{q}}$$

$$= \sup_{B=B(x,r)} \left(\left(\frac{1}{r^{\lambda}}\right)^{\frac{1}{q}} \left(\int_{B(x,r)} |f(x)|^q \, dx\right)^{\frac{1}{q}}\right)$$

Because $\left(\frac{1}{r^{\lambda}}\right)^{\frac{1}{q}}$ is a constant, it can be written as follows :

$$= \left(\frac{1}{r^{\lambda}}\right)^{\frac{1}{q}} \sup_{B(x,r)} \left(\int_{B(x,r)} |f(x)|^q \, dx\right)^{\frac{1}{q}}$$

$$= \frac{1}{r^{\frac{\lambda}{q}}} \sup_{B(x,r)} \left(\int_{B(x,r)} |f(x)|^q \, dx\right)^{\frac{1}{q}}$$

Because $\|f\|_{q,\lambda} < \infty$, then it should $\left\{\left(\frac{1}{r^{\lambda}} \int_{B(x,r)} |f(x)|^q \, dx\right)^{\frac{1}{q}} \middle| x \in \mathbb{R}^n\right\}$ exists and is exists and is limited $\exists C > 0$ such that $\left(\frac{1}{r^{\lambda}} \int_{B(x,r)} |f(x)|^q \, dx\right)^{\frac{1}{q}} < C, \forall x \in \mathbb{R}^n$ or as $\int_{B(x,r)} |f(x)|^q \, dx < Cr^{\lambda}, \forall x \in \mathbb{R}^n.$ So if $f \in L^{q,\lambda}(\mathbb{R}^n)$ so $\exists C > 0 \ni$

For example $\int_{B(x,r)} |f(x)|^q \, dx < Cr^{\lambda}, \forall\, x \in \mathbb{R}^n$ and $r > 0$. Hence, it can be written become :

$$\frac{1}{r^{\lambda}} \int_{B(x,r)} |f(x)|^q \, dx < C, \forall\, x \in \mathbb{R}^n.$$

This shows $\frac{1}{r^\lambda}\int_{B(x,r)}|f(x)|^q\,dx$ limited $,\forall\, x\in\mathbb{R}^n$. Because it's limited has a supremum value with $\sup\limits_{B(x,r)}\left(\frac{1}{r^\lambda}\int_{B(x,r)}|f(x)|^q\,dx\right)^{\frac{1}{q}}<\infty$. Because it applies $\forall x\in\mathbb{R}^n$ so $\sup\limits_{B(x,r)}\left(\frac{1}{r^\lambda}\int_{B(x,r)}|f(x)|^q\,dx\right)^{\frac{1}{q}}=\frac{1}{r^\lambda}\int_{B(x,r)}|f(x)|^q\,dx<C<\infty$. Proved that $f\in L^{q,\lambda}(\mathbb{R}^n)$. ■

Therefore, $f\in L^{q,\lambda}(\mathbb{R}^n)$ if and only if $\exists\, C>0\ni\int_{B(x,r)}|f(x)|^q dx<Cr^\lambda\,\forall\, x\in\mathbb{R}^n$ dan $r>0$. This can be interpreted that the rate of growth $\int_{B(x,r)}|f(x)|^q dx$ controlled by the expression on the right side ie r^λ, with λ is called the order of f.

Besides the Morrey room $L^{q,\lambda}(\mathbb{R}^n)$ defined above, there are other variants of $L^{q,\lambda}(\mathbb{R}^n)$ yakni ruang Morrey $\mathcal{M}_q^p(\mathbb{R}^n)$ which is more often called the classic Morrey space. Dalam As of this writing, the more studied Morrey space is the classical Morrey space. The study that is interesting to discuss is how the nature of the limited space.

2. Classic Morrey Room $\boldsymbol{\mathcal{M}_q^p}(\mathbb{R}^{\boldsymbol{n}})$

$\mathcal{M}_q^p(\mathbb{R}^n)$ is one variation of $L^{q,\lambda}(\mathbb{R}^n)$ which has the following definition:

Definition 1. Classical Morrey Room $\boldsymbol{\mathcal{M}_q^p}(\mathbb{R}^{\boldsymbol{n}})$

For $1\le q\le p<\infty$, *Morrey room* $\mathcal{M}_q^p(\mathbb{R}^n)$ *is the set of all functions* $f\in L^q_{loc}(\mathbb{R}^n)$ *that fulfills* $\|f\|_{\mathcal{M}_q^p}<\infty$ *where the norm is defined as follows :*

$$\|f\|_{\mathcal{M}_q^p}=\sup_{\substack{a\in\mathbb{R}^n\\ r>0}}|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}\left[\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}}$$

With $|B(a,r)|$ Specifies the size (Lebesgue) of the ball $B(a,r)$. Here is the relationship between the classic Morrey spaces $\mathcal{M}_q^p(\mathbb{R}^n)$ with Morrey's room $L^{q,\lambda}(\mathbb{R}^n)$ sebagaimana presented in the following proposition :

Proposition 1. Relations between $\boldsymbol{\mathcal{M}_q^p}(\mathbb{R}^{\boldsymbol{n}})$ with $\boldsymbol{L^{q,\lambda}}(\mathbb{R}^{\boldsymbol{n}})$

$$\mathcal{M}_q^p(\mathbb{R}^n)=L^{q,n\left(1-\frac{q}{p}\right)}(\mathbb{R}^n)$$

Proof:

Take any function $f\in\mathcal{M}_q^p(\mathbb{R}^n)$ so $f\in L^q_{loc}(\mathbb{R}^n)$ with $\|f\|_{\mathcal{M}_q^p}<\infty$. Therefore $\forall B(a,r)\subseteq\mathbb{R}^n$,

$$|B(a,r)|^{\left(\frac{1}{p}-\frac{1}{q}\right)}\left[\iint_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} < \infty$$

$|B(a,r)|$ stated lebesgue size, it means $|B(a,r)| = Cr^n$ for a $C > 0$.

$$|B(a,r)|^{\left(\frac{1}{p}-\frac{1}{q}\right)}\left[\iint_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} \dots \ (*)$$

$$= cr^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\left[\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} = \frac{c^{\frac{1}{p}} r^{\frac{n}{p}}}{c^{\frac{1}{q}} r^{\frac{n}{q}}}\left[\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}}$$

$$= \frac{c^{\frac{1}{p}}}{c^{\frac{1}{q}}} r^{\frac{n}{p}}\left[\frac{1}{r^n}\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} = \frac{c^{\frac{1}{p}}}{c^{\frac{1}{q}}}\frac{1}{\left(r^{\frac{-n}{p}}\right)}\left[\frac{1}{r^n}\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}}$$

$$= \frac{c^{\frac{1}{p}}}{c^{\frac{1}{q}}}\left(\frac{1}{\left(r^{-\frac{n}{p}}\right)^q}\right)^{\frac{1}{q}}\left[\frac{1}{r^n}\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} = \frac{c^{\frac{1}{p}}}{c^{\frac{1}{q}}}\left[\frac{1}{r^{-\frac{nq}{p}}}\frac{1}{r^n}\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}}$$

$$= \frac{c^{\frac{1}{p}}}{c^{\frac{1}{q}}}\left[\frac{1}{\left(r^{-\frac{n}{p}}\right)^q}\frac{1}{r^n}\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} = \frac{c^{\frac{1}{p}}}{c^{\frac{1}{q}}}\left[\frac{1}{r^{n\left(1-\frac{q}{p}\right)}}\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}}.$$

Based on $(*)$ then obtained:

$\left[\frac{1}{r^{n\left(1-\frac{q}{p}\right)}}\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} < \infty$, for a $B(a,r) \subseteq \mathbb{R}^{\boldsymbol{n}}$. Thus, when take the supremum on the left side for each $B(a,r) \subseteq \mathbb{R}^{\boldsymbol{n}}$, obtained :

$$\|f\|_{L^{q,n\left(1-\frac{q}{p}\right)}} = \sup_{B(x,r)}\left[\frac{1}{r^{n\left(1-\frac{q}{p}\right)}}\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} < \infty$$

as a result, $f \in L^{q,n\left(1-\frac{q}{p}\right)}(\mathbb{R}^{\boldsymbol{n}})$.

Instead, take any function $f \in L^{q,n\left(1-\frac{q}{p}\right)}(\mathbb{R}^n)$. Hence for every $B(x,r)\subseteq\mathbb{R}^n$, apply :

$$\|f\|_{L^{q,n\left(1-\frac{q}{p}\right)}} = \sup_{B(x,r)}\left[\frac{1}{r^{n\left(1-\frac{q}{p}\right)}}\int_{B(x,r)}|f(x)|^q dx\right]^{\frac{1}{q}} < \infty$$

then, for each $B(x,r)\subseteq\mathbb{R}^n$, obtained:

$$\left[\frac{1}{r^{n\left(1-\frac{q}{p}\right)}}\int_{B(x,r)}|f(x)|^q dx\right]^{\frac{1}{q}} < \infty$$

Noted that : $\left(r^{n\left(1-\frac{q}{p}\right)}\right)^{\frac{1}{q}} = r^{n\left(\frac{1}{q}-\frac{q}{pq}\right)} = r^{n\left(\frac{1}{q}-\frac{1}{p}\right)}$, so that

$$\left[\frac{1}{r^{n\left(1-\frac{q}{p}\right)}}\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} = r^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\left[\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}}$$

$$= |B(a,r)|^{\frac{1}{p}-\frac{1}{q}}\left[\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} < \infty.$$

as a result, $|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}\left[\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} < \infty$. This applies to every $B(a,r)\subseteq\mathbb{R}^n$, so that when taking the supremum on the left side it is obtained :

$$\|f\|_{\mathcal{M}_q^p} = \sup_{\substack{a\in\mathbb{R}^n\\ r>0}}|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}\left[\frac{1}{r^{n\left(1-\frac{q}{p}\right)}}\int_{B(a,r)}|f(x)|^q dx\right]^{\frac{1}{q}} < \infty$$

as a result, $f \in \mathcal{M}_q^p(\mathbb{R}^n)$. ■

Therefore, $\mathcal{M}_q^p(\mathbb{R}^n) = L^{q,n\left(1-\frac{q}{p}\right)}(\mathbb{R}^n)$. Based on the above proposition, the following results are obtained :

Consequences 1. Relations between $\mathcal{M}_q^p(\mathbb{R}^n)$ with $L^{q,\lambda}(\mathbb{R}^n)$

$$L^{q,\lambda}(\mathbb{R}^n) = \mathcal{M}_q^{\frac{nq}{n-\lambda}}(\mathbb{R}^n)$$

To prove the consequence of the above proposition, suppose $\lambda = n\left(1-\frac{q}{p}\right)$. Based on the previous proposition, $\mathcal{M}_q^p(\mathbb{R}^n) = L^{q,\lambda}(\mathbb{R}^n)$. By constructing $\lambda = n\left(1-\frac{q}{p}\right)$, such that it will be obtained: $p = \frac{nq}{n-\lambda}$. Furthermore, by using the same method in the above proposition, we will obtain a relation such as Effect 1. on.

Results and Discussion

Theorem 1.

If $f \in L^q(\mathbb{R}^n)$ *with* $1 < q \leq \infty$, *so* $Mf \in L^q(\mathbb{R}^n)$ *And*

$$\|Mf\|_q \leq A_q \|f\|_q$$

with A_q *is a positive constant that depends only on* q *and* n.

Evidence: For the case $q = \infty$ trivia with $A_\infty = 1$, because the essential supremum of a function will not be less than the average value of the function. Now assume for case $1 < q < \infty$. To prove this theorem, it is necessary to define a function as follows:

$$f_1(x) = \begin{cases} f(x), & jika\ |f(x)| \geq \frac{\lambda}{2} \\ 0, & lainnya \end{cases}.$$

Since the form of the function is as above, it is obtained

$$|f(x)| \leq |f_1(x)| + \tfrac{\lambda}{2} \text{ dan } |Mf(x)| \leq |Mf_1(x)| + \tfrac{\lambda}{2}$$

as well as

$$\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \subset \left\{x \in \mathbb{R}^n : Mf(x) > \frac{\lambda}{2}\right\}$$

by using if $f \in L^q(\mathbb{R}^n)$, so $f_1 \in L^1(\mathbb{R}^n)$ such that it is obtained

$$m(E_\lambda) = m\{x \in \mathbb{R}^n : Mf(x) > \lambda\}$$

$$\leq 2\frac{A_n}{\lambda}\|f_1\|_1 = 2\frac{A_n}{\lambda}\int_{x \in \mathbb{R}^n : |f(x)| \geq \frac{\lambda}{2}} |f(x)|dx \ldots (1)$$

Now following the definition of the maximal operator Hardy–Littlewood M, for example $g = Mf$ and μ is the distribution function of g as well as using the partial integral technique, obtained

$$\int_{\mathbb{R}^n} (Mf)^q dx = -\int_0^\infty \lambda^q d\,\mu(\lambda) = q\int_0^\infty \lambda^{q-1}\mu(\lambda)d\,\lambda$$

By using the equation (1), so

$$\|Mf\|_q^q = q\int_0^\infty \lambda^{q-1} m(E_\lambda)d\,\lambda \leq q\int_0^\infty \lambda^{q-1}\left(2\frac{A_n}{\lambda}\int_{x \in \mathbb{R}^n : |f(x)| \geq \frac{\lambda}{2}} |f(x)|dx\right)d\lambda.$$

The fold integral above is solved by changing the order of its integration, namely integrating it first against λ so that the integral results are obtained which are as follows:

$$\int_0^{2|f(x)|} \lambda^{q-2} d\,\lambda = \frac{1}{q-1}|2f(x)|^{q-1}.$$

So, the fold integral above has value

$$2\frac{A_n q}{q-1}\int_{x\in\mathbb{R}^n}|f(x)|\,|2f(x)|^{q-1}dx = 4\frac{A_n q}{q-1}\int_{x\in\mathbb{R}^n}|f(x)|^q dx = A_q^q\|f\|_q^q$$

so that $\|Mf\|_q^q \leq A_q^q\|f\|_q^q$ or in other words $\|Mf\|_q \leq A_q\|f\|_q$, with A_q is a constant that only depends on q And n. ■

Theorem 1. which we just proved says that the operator is maximal Hardy–Littlewood M confined to the Lebesgue space $L^q(\mathbb{R}^n)$. With the help of the Theorem 1. this will prove the limitation of the maximum operator Hardy-Littlewood M in a classic Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ which is presented in the following theorem:

Theorem 2.

If $f \in \mathcal{M}_q^p(\mathbb{R}^n)$, *with* $1 < q \leq p < \infty$ *so*

$$\|Mf\|_{\mathcal{M}_q^p(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}_q^p(\mathbb{R}^n)}.$$

Proof:

Take any function $f \in \mathcal{M}_q^p(\mathbb{R}^n)$ and balls $B = B(a,r) \subset \mathbb{R}^n$. For example $f = f_1 + f_2$ with the definition of the function as follows:

$$f_1(x) = \begin{cases} f(x), jika\ x \in 5B = B(a,5r) \\ 0, \qquad lainnya \end{cases}$$

dan $$f_2(x) = \begin{cases} f(x), jika\ x \notin 5B = B(a,5r) \\ 0, \qquad lainnya \end{cases}$$

Note that

$$|B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{B=B(a,r)} Mf_1(t)^q dt\right)^{\frac{1}{q}} \leq |B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\mathbb{R}^n} Mf_1(t)^q dt\right)^{\frac{1}{q}}$$

$$\lesssim |B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{5B} f_1(t)^q dt\right)^{\frac{1}{q}}$$

$$\lesssim |5B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{5B} f_1(t)^q dt\right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_{\mathcal{M}_q^p} \qquad\qquad (*)$$

Next notice that if R is a cutting ball B dan $\mathbb{R}^n\backslash B$ hence the diameters $(R) \geq 2$ diameter(B) And $2R \supset B$. Therefore,

$$Mf_2(t) \lesssim \sup_{B\subset\mathbb{R}} \frac{1}{|R|}\int_R |f(t)|dt.$$

As a result,

$$|B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_B Mf_2(t)^q dt\right)^{\frac{1}{q}} \lesssim |B|^{\frac{1}{p}} \sup_{B\subset\mathbb{R}} \frac{1}{|R|}\int_R |f(t)|dt$$

such that

$$|B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_B Mf_2(t)^q dt\right)^{\frac{1}{q}} \lesssim \sup_{\mathbb{R}}|R|^{\frac{1}{p}-1}\int_R |f(t)|dt = \|f\|_{\mathcal{M}_1^p} \leq \|f\|_{\mathcal{M}_q^p} \ldots (**)$$

Based on $(*)$ And $(**)$ obtained : $|B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{B=B(a,r)} Mf_1(t)^q dt\right)^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}_q^p}$

And $|B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{B=B(a,r)} Mf_2(t)^q dt\right)^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}_q^p}$.

as a result, $|B|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{B=B(a,r)} Mf(t)^q dt\right)^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}_q^p}$. By taking the supremum on the left-hand side for all balls $B = B(a,r) \subseteq \mathbb{R}^n$, obtained $\|Mf\|_{\mathcal{M}_q^p} \lesssim \|f\|_{\mathcal{M}_q^p}$. ■

Fractional Integral Operator I_α

One of the operators that can also be seen in the Lebesgue space $L^q(\mathbb{R}^n)$ is the fractional integral operator I_α which maps functions to other functions on $\mathbb{R}^n$. These operators are widely used in the field of Fourier Analysis, integral equation, as well as partial differential equations. Here is the definition of the fractional integral operator.

Definition 1. Fractional Integral Operator I_α

For $0 < \alpha < n$ *And* $x \in \mathbb{R}^n$, *fractional integral operator* $\boldsymbol{I_\alpha}$ *defined as follows:*

$$\boldsymbol{I_\alpha} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

Where f *is a real-valued function on* $\mathbb{R}^n$.

Fractional integral operator I_α often known as Potential Riesz (by degrees α). If in case $\alpha = 2$, operator I_α often referred to as the Newtonian potential. This operator was first studied by Hardy-Littlewood (1927) and by Sobolev (1938). In the book, Hardy-Littlewood proves that I_α is the operator that carries the function in $L^q(\mathbb{R}^n)$ ke $L^s(\mathbb{R}^n)$. This is explained in more detail in the following theorem:

Theorem 3. Hardy-Littlewood-Sobolev inequality

$If\ \alpha = \frac{n}{q} - \frac{n}{s}, 1 < q < s < \infty, so\ \|I_\alpha f\|_s \lesssim \|f\|_q.$

Proof:

Write

$$I_\alpha f(x) = \int_{|x-y|<R} \frac{f(y)}{|x-y|^{n-\alpha}} dy + \int_{|x-y|\geq R} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$
$$= I_1 f(x) + I_2 f(x).$$

Note that $I_1 f(x)$ can be approximated as follows :

$$|I_1 f(x)| \leq \int_{|x-y|<R} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$
$$\leq \sum_{k=-\infty}^{1} \int_{B(x,2^{k+1}R)\backslash B(x,2^k R)} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

By using that fact $|x-y| \geq 2^k R$, then obtained:

$$\leq \sum_{k=-\infty}^{1} \frac{1}{(2^k R)^{n-\alpha}} \int_{B(x,2^{k+1}R)\backslash B(x,2^k R)} |f(y)| dy$$
$$\leq \sum_{k=-\infty}^{1} \frac{1}{(2^k R)^{n-\alpha}} \int_{B(x,2^{k+1}R)} |f(y)| dy$$
$$\lesssim \sum_{k=-\infty}^{1} \frac{1}{(2^k R)^{n-\alpha}} \left((2^{k+1}R)^n Mf(x)\right)$$
$$\lesssim 2^n \sum_{k=-\infty}^{1} (2^\alpha)^k R^\alpha Mf(x) \lesssim R^\alpha Mf(x).$$

So obtained : $|I_1 f(x)| \lesssim R^\alpha Mf(x).$ $(*)$

In approximating $I_2(x)$ inequality is used Hölder. Note that :

$$|I_2 f(x)| \leq \int_{|x-y|\geq R} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

$$\leq \sum_{k=0}^{\infty} \int_{B(x,2^{k+1}R)\setminus B(x,2^kR)} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

$$\leq \sum_{k=0}^{\infty} \frac{1}{(2^kR)^{n-\alpha}} \int_{B(x,2^{k+1}R)\setminus B(x,2^kR)} |f(y)| dy$$

$$\leq \sum_{k=0}^{\infty} \frac{1}{(2^kR)^{n-\alpha}} \int_{B(x,2^{k+1}R)} |f(y)| dy$$

$$\lesssim \sum_{k=0}^{\infty} \frac{1}{(2^kR)^{n-\alpha}} \left[\int_{B(x,2^{k+1}R)} |f(y)|^q dy\right]^{\frac{1}{q}} \left[\int_{B(x,2^{k+1}R)} dy\right]^{1-\frac{1}{q}}$$

$$\lesssim \sum_{k=-\infty}^{1} \frac{(2^{k+1}R)^{n\left(1-\frac{1}{q}\right)}}{(2^kR)^{n-\alpha}} \left[\int_{B(x,2^{k+1}R)} |f(y)|^q dy\right]^{\frac{1}{q}} \lesssim R^{-\frac{n}{s}} \|f\|_q.$$

The result is obtained : $|I_2 f(x)| \lesssim R^{-\frac{n}{s}} \|f\|_q.$ $(**)$

Based on $(*)$ And $(**)$ obtained :

$$|I_\alpha f(x)| \leq |I_1 f(x)| + |I_2 f(x)|$$

$$\leq C\left(R^\alpha Mf(x) + R^{-\frac{n}{s}} \|f\|_q\right). \qquad (***)$$

For $R > 0$, selectable R such that

$$\frac{Mf(x)}{\|f\|_q} = R^{-\frac{n}{s}-\alpha} = R^{-\frac{n}{q}}. \qquad (1)$$

So that if R is selected as in (1), then equation $(***)$ become

$$|I_\alpha f(x)| \lesssim \left(Mf(x)\right)^{\frac{q}{s}} \|f\|_q^{1-\frac{q}{s}}.$$

as a result

$$\|I_\alpha f(x)\|_s^s = \int_{\mathbb{R}^n} |I_\alpha f(y)|^s dy$$

$$\lesssim \int_{\mathbb{R}^n} \left(Mf(y)\right)^q \|f\|_q^{s-q} dy$$

$$\lesssim \|f\|_q^{s-q} \int_{\mathbb{R}^n} \left(Mf(y)\right)^q dy$$

By making use of the theorem on 3.5.3 that is $\|Mf(x)\|_q \lesssim \|f\|_q$, then obtained :

$$\lesssim \|f\|_q^{s-q}\|f\|_q^q \lesssim \|f\|_q^s.$$

Therefore, it is proved that $\|I_\alpha f(x)\|_s \lesssim \|f\|_q$. ■

Theorem 3. has proved that it turns out that the integral operator is fractional I_α confined to the Lebesgue space, it will now be shown that the operator I_α also limited to classic Morrey room. Operators limitations I_α in Morrey's room it was first studied by Adam (1975) and Chiarenza, F. – M. Frasca (1987). Evidence of carrier limitations I_α in the classical Morrey space is not much different from the proof in space Lebesgue. In addition to exploiting the limited nature of the operator I_α in the Lebesgue room, also the limiting nature of the maximal operator Hardy–Littlewood M in the classical Morrey space is used in the proof.

Theorem 4.

For example given $1 < q \leq p < \frac{n}{\alpha}$ *and* $0 < \alpha < n$. *If for* $1 < t \leq s < \infty$ *apply*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n} \ And \frac{q}{p} = \frac{t}{s}$$

so

$$\|I_\alpha f(x)\|_{\mathcal{M}_t^s} \lesssim \|f\|_{\mathcal{M}_q^p}.$$

Proof:

Write back

$$I_\alpha f(x) = \int_{|x-y|<R} \frac{f(y)}{|x-y|^{n-\alpha}} dy + \int_{|x-y|\geq R} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$
$$= I_1 f(x) + I_2 f(x).$$

Note that $I_1 f(x)$ can be approximated as follows:

$|I_1 f(x)| \leq \int_{|x-y|<R} \frac{f(y)}{|x-y|^{n-\alpha}} dy$

$$\leq \sum_{k=-\infty}^{1} \int_{B(x,2^{k+1}R)\setminus B(x,2^k R)} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

By using that fact $|x-y| \geq 2^k R$, then obtained:

$$\leq \sum_{k=-\infty}^{1} \frac{1}{(2^k R)^{n-\alpha}} \int_{B(x,2^{k+1}R)\setminus B(x,2^k R)} |f(y)| dy$$

$$\leq \sum_{k=-\infty}^{1} \frac{1}{(2^k R)^{n-\alpha}} \int_{B(x,2^{k+1}R)} |f(y)| dy$$

$$\lesssim \sum_{k=-\infty}^{1} \frac{1}{(2^k R)^{n-\alpha}} \left((2^{k+1}R)^n Mf(x)\right)$$

$$\lesssim 2^n \sum_{k=-\infty}^{1} (2^\alpha)^k R^\alpha Mf(x)$$

$$\lesssim R^\alpha Mf(x).$$

So obtained : $|I_1 f(x)| \lesssim R^\alpha Mf(x).$ $(*)$

In approximating $I_2(x)$ reuse inequality H ö lder. Note that : $|I_2 f(x)| \leq \int_{|x-y|\geq R} \frac{f(y)}{|x-y|^{n-\alpha}} dy$

$$\leq \sum_{k=0}^{\infty} \int_{B(x,2^{k+1}R)\setminus B(x,2^k R)} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

$$\leq \sum_{k=0}^{\infty} \frac{1}{(2^k R)^{n-\alpha}} \int_{B(x,2^{k+1}R)\setminus B(x,2^k R)} |f(y)| dy$$

$$\leq \sum_{k=0}^{\infty} \frac{1}{(2^k R)^{n-\alpha}} \int_{B(x,2^{k+1}R)} |f(y)| dy$$

$$\lesssim \sum_{k=0}^{\infty} \frac{1}{(2^k R)^{n-\alpha}} \left[\int_{B(x,2^{k+1}R)} |f(y)|^q dy\right]^{\frac{1}{q}} \left[\int_{B(x,2^{k+1}R)} dy\right]^{1-\frac{1}{q}}$$

$$\lesssim \sum_{k=0}^{\infty} \frac{(2^{k+1}R)^{n\left(1-\frac{1}{q}\right)}}{(2^k R)^{n-\alpha}} \left[\int_{B(x,2^{k+1}R)} |f(y)|^q dy\right]^{\frac{1}{q}}$$

Because

$$|B(x,2^{k+1}R)|^{\frac{1}{p}-\frac{1}{q}} \left[\int_{B(x,2^{k+1}R)} |f(y)|^q dy\right]^{\frac{1}{q}} \leq \|f\|_{\mathcal{M}_q^p},$$

then obtained

$$\left[\int_{B(x,2^{k+1}R)}|f(y)|^q dy\right]^{\frac{1}{q}} \le \frac{\|f\|_{\mathcal{M}_q^p}}{|B(x,2^{k+1}R)|^{\frac{1}{p}-\frac{1}{q}}} \le \frac{\|f\|_{\mathcal{M}_q^p}}{(2^{k+1}R)^{n\left(\frac{1}{p}-\frac{1}{q}\right)}}$$

As a result,

$$|I_2 f(x)| \lesssim \sum_{k=0}^{\infty} \frac{(2^{k+1}R)^{n\left(1-\frac{1}{q}\right)}}{(2^k R)^{n-\alpha}} \left[\int_{B(x,2^{k+1}R)}|f(y)|^q dy\right]^{\frac{1}{q}}$$

$$\lesssim \sum_{k=0}^{\infty} \frac{(2^{k+1}R)^{n\left(1-\frac{1}{q}\right)}}{(2^k R)^{n-\alpha}} \frac{\|f\|_{\mathcal{M}_q^p}}{(2^{k+1}R)^{n\left(\frac{1}{p}-\frac{1}{q}\right)}} \lesssim R^{-\frac{n}{s}}\|f\|_{\mathcal{M}_q^p}.$$

The result is obtained : $|I_2 f(x)| \lesssim\lesssim R^{-\frac{n}{s}}\|f\|_{\mathcal{M}_q^p}.$ $(**)$

Based on $(*)$ And $(**)$ obtained :

$$|I_\alpha f(x)| \le |I_1 f(x)| + |I_2 f(x)|$$

$$\le C\left(R^\alpha Mf(x) + R^{-\frac{n}{s}}\|f\|_q\right). \qquad (***)$$

For R>0, we can choose R such that

$$R^\alpha Mf(x) = R^{-\frac{n}{s}}\|f\|_{\mathcal{M}_q^p}$$

or

$$R = \left(\frac{Mf(x)}{\|f\|_{\mathcal{M}_q^p}}\right)^{\frac{p}{n}} \qquad (1)$$

So that if R is selected as in (1), then equation $(***)$ become

$$|I_\alpha f(x)| \lesssim |Mf(x)|^{\frac{p}{s}}\|f\|_{\mathcal{M}_q^p}^{1-\frac{p}{s}} = |Mf(x)|^{\frac{q}{t}}\|f\|_{\mathcal{M}_q^p}^{1-\frac{p}{s}}$$

On the other hand,

$$|B(x,r)|^{\frac{1}{p}-\frac{1}{q}}\left[\int_{B(x,r)}|Mf(x)|^q dx\right]^{\frac{1}{q}} \le \|Mf\|_{\mathcal{M}_q^p},$$

so that

$$\left[\int_{B(x,r)}|Mf(x)|^q dx\right]^{\frac{1}{q}} \le \frac{\|Mf\|_{\mathcal{M}_q^p}}{|B(x,r)|^{\frac{1}{p}-\frac{1}{q}}}$$

or it can be written as follows :

$$\left[\int_{B(x,r)}|Mf(x)|^q dx\right]^{\frac{1}{t}} \leq \frac{\|Mf\|^{\frac{q}{t}}_{\mathcal{M}_q^p}}{|B(x,r)|^{\frac{1}{t}\left(\frac{q}{p}-1\right)}} = \frac{\|Mf\|^{\frac{q}{t}}_{\mathcal{M}_q^p}}{|B(x,r)|^{\frac{1}{t}\left(\frac{t}{s}-1\right)}} = \frac{\|Mf\|^{\frac{q}{t}}_{\mathcal{M}_q^p}}{|B(x,r)|^{\frac{1}{s}-\frac{1}{t}}}.$$

Hence, obtained

$$\|I_\alpha f(x)\|_{\mathcal{M}_t^s} = \sup_{\substack{x\in\mathbb{R}^n\\ r>0}} |B(x,r)|^{\frac{1}{s}-\frac{1}{t}}\left[\int_{B(x,r)}|I_\alpha f(x)|^t dx\right]^{\frac{1}{t}}$$

$$\lesssim \sup_{\substack{x\in\mathbb{R}^n\\ r>0}} |B(x,r)|^{\frac{1}{s}-\frac{1}{t}}\left[\int_{B(x,r)}|Mf(x)|^q \|f\|^{t(1-\frac{p}{s})}_{\mathcal{M}_q^p} dx\right]^{\frac{1}{t}}$$

$$\lesssim \sup_{\substack{x\in\mathbb{R}^n\\ r>0}} |B(x,r)|^{\frac{1}{s}-\frac{1}{t}}\|f\|^{(1-\frac{p}{s})}_{\mathcal{M}_q^p}\left[\int_{B(x,r)}|Mf(x)|^q dx\right]^{\frac{1}{t}}$$

$$\lesssim \sup_{\substack{x\in\mathbb{R}^n\\ r>0}} |B(x,r)|^{\frac{1}{s}-\frac{1}{t}}\|f\|^{(1-\frac{p}{s})}_{\mathcal{M}_q^p}\frac{\|Mf\|^{\frac{q}{t}}_{\mathcal{M}_q^p}}{|B(x,r)|^{\frac{1}{s}-\frac{1}{t}}}$$

$$\lesssim \sup_{\substack{x\in\mathbb{R}^n\\ r>0}} \|f\|^{(1-\frac{p}{s})}_{\mathcal{M}_q^p}\frac{\|Mf\|^{\frac{q}{t}}_{\mathcal{M}_q^p}}{|B(x,r)|^{\frac{1}{s}-\frac{1}{t}}}$$

$$\lesssim \|f\|^{(1-\frac{p}{s})}_{\mathcal{M}_q^p}\|Mf\|^{\frac{p}{s}}_{\mathcal{M}_q^p}$$

By using the limited property of the maximum operator Hardy-Littlewood M in the classic Morrey space it has been proven that is $\|Mf\|_{\mathcal{M}_q^p(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}_q^p(\mathbb{R}^n)}$, then obtained:

$$\lesssim \|f\|^{(1-\frac{p}{s})}_{\mathcal{M}_q^p}\|Mf\|^{\frac{p}{s}}_{\mathcal{M}_q^p} \lesssim \|f\|^{(1-\frac{p}{s})}_{\mathcal{M}_q^p}\|f\|^{\frac{p}{s}}_{\mathcal{M}_q^p} = \|f\|_{\mathcal{M}_q^p}.$$

as a result,

$$\|I_\alpha f(x)\|_{\mathcal{M}_t^s} \lesssim \|f\|_{\mathcal{M}_q^p}. \quad \blacksquare$$

Look at the final result of the Theorem 4. such, if $p = q$ so $s = t$, means theorem 4. above is the Hardy-Littlewood-Sobolev Inequality. Thus, it has been shown that the integral operator is fractional I_α has limited properties in the Lebesgue space $L^q(\mathbb{R}^n)$ and in the classic Morrey room $\mathcal{M}_q^p(\mathbb{R}^n)$.

Conclusion

Based on the discussion in the previous chapter, the following conclusions are obtained: 1) Morrey Room $L^{q,\lambda}(\mathbb{R}^n)$ is an expansion (refinement) of the Lebesgue space $L^q(\mathbb{R}^n)$, especially for cases $\lambda = 0, L^{q,\lambda}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$. Classic Morrey Room$\mathcal{M}_q^p(\mathbb{R}^n)$ is a normed space and a Banach space. 2) The fractional integral operator I_\alpha has a similar limitation to the Lebesgue space $L^q(\mathbb{R}^n)$ and the classic Morrey room$\mathcal{M}_q^p(\mathbb{R}^n)$.

BIBLIOGRAFI

Adams, D, R. (1975). *A note on Riesz potential*. *Duke Math. J.*42, 765-778.

Barttle, R. G., & Sherbert, D. R. (2000). *Introduction to Real Analysis* (3rd ed.).

New York: John Wiley & Sons.

Barttle, R. G., & Sherbert, D. R. (1966). *Elements of Integration and Lebesgue Measure* New York: John Wiley & Sons.

Chiarenza, F. dan M. Frasca. (1987). *Morrey spaces and Hardy-Littlewood maximal function.* Rend. Mat. 7, 273-279.

Eridani (2005). *Ruang Morrey dan operator integral fraksional.* Desertasi. ITB Bandung.

Gunawan, H. (2003). *A note on the generalized fractional integral operators*. J. Indones. Math. Soc. (MIHMI), 9:1, 39-43.

Gupta, V. P., & Jain, P. K. (1986). *Lebesgue Measure and Integration.* New Delhi: John Wiley & Sons.

Hardy, G. H. dan J. E. Littlewood. (1927). *Some properties of fractional integral I*. Math.Zeit., 27, 565-606.

Ismail (2007). *Ruang Lebesgue.* Skripsi. UGM Yogyakarta.

Kreyszig. (1978). *Introductory Functional Analysis With Application.* Canada:

John Wiley & Sons.

Kufner, A., John, O., & Fucik, S. (1977). *Function Spaces.* Czechoslovakia:

Noordhoff International Publishing.

Limanta M., Kevin. (2014). *Ruang Morrey kuat dan ruang Morrey lemah.*

Skripsi. ITB Bandung.

Masta, A. (2011). *Karakteristik Integral McShane di Ruang Euclid* $(\mathbb{R}^n)$. Skripsi,

UPI Bandung.

Nurhayati, Lina (2013). *Keterbatasan operator integral fraksional diperumum di ruang Morrey yang diperumum. Ruang Morrey kuat dan ruang Morrey lemah.* Tesis. ITB Bandung.

Royden, H. L., & Fitzpatrick, P. M. (1988). *Real Analysis.* London:

China Machine Press.

Stein, E. M. (1970). *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press: New Jersey.

Stein, E. M. (1993). *Harmonic Analysis: Real Variable Methods, Orthogonality, and*

Oscillatory Integrals. Princeton University Press: New Jersey.

Universitas Pendidikan Indonesia. (2013). *Pedoman Penulisan Karya Tulis Ilmiah.*

Bandung: UPI Press.

Utari, Mila. (2014). *Refleksifitas pada Ruang Orlicz dengan kekonvergenan rata-rata.* Skripsi. UPI Bandung.

Wayne. (1973). *Convex Function.* New York: Akademic Press.